Multiple and last-minute bidding in competing Internet auctions

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Abstract

The present paper proposes a theoretical model of the optimal bidding behavior in simultaneous competing Internet auctions, which are very popular on eBay. We argue that in order to avoid the risk of paying too high a price, bidders wish to learn how valuable the good appears to the other bidders and basing on that information to allocate between the auctions. To do so, they prefer to first bid less aggressively to test the market and next to bid more aggressively in the auction chosen according to what they learn from the market. We also show that in an equilibrium, bidders may bid in the last-minute, even-thought last-minute bidding causes inefficiency. Finally, we indicate that the competition between the auctions introduces certain asymmetry in optimal bidding strategies.

Keywords: Internet auctions, eBay, multi-unit auctions, simultaneous auctions, last-minute bidding, sniping
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1 Introduction

eBay gives an opportunity to buy popular goods in many simultaneous auctions.[1] On eBay, identical goods are not only offered by private sellers, but also by online stores. The same goods may be also often bought in ordinary stores.

The presence of multiple auctions offering identical products should have an impact on bidders' behavior. The number of auctions may be interpreted as the level of supply. As long as demand is elastic, the increased supply normally causes prices to fall. Hence, it seems reasonable to expect that bidders are less aggressive, when the number of auctions increases.

The existing literature on Internet auctions tend to treat each Internet auction in separation from the other Internet auctions offering identical goods. In particular, the economic studies of Internet auctions have mainly focused on single-unit auctions, which may be interpreted as Internet auctions selling unique goods or Internet auctions selling the same goods but each time to a

[1] See pulse.eBay.com for a list of the most popular searches.

different group of bidders (see for example Bajari and Hortacsu, 2003 and 2004, and Roth and Ockenfels, 2002 and 2006).

Peters and Severinov (2006) analyzed competing Internet auctions, but decided to focus on different ending rule than the one used on eBay. In their model identical items may be bough in several dynamic auctions that differ from auctions on eBay in how they end. While eBay opts for a fixed ending rule according to which an auction ends at the time specified in advance, the model studied by Peters and Severinov allows auctions to continue until all bidders pass.

The fact that Peters and Severinov (2006) assume a soft-close ending rule seems to be an important misrepresentation of Internet auctions on eBay, as the ending rule is known to have an impact on bidders' behavior. Roth and Ockenfels (2002) empirically showed that in hard-close Internet auctions, bidders bid later than in soft-close Internet auctions. This phenomenon was also observed in the laboratory experiment conducted by Ariely, Ockenfels and Roth (2005) and theoretically explained by Roth and Ockenfels (2006).

The present paper proposes a theoretical model of two competing hard-close Internet auctions and proves that the optimal behavior involves last-minute bidding and multiple bidding, the two patterns commonly observed on eBay (see Bajari and Hortacsu, 2003, and Roth and Ockenfels, 2002 and 2006). The phenomenon of last-minute bidding (that is bidding in the very last minute of the auction) has been also discussed by Roth and Ockenfels (2002), who indicated that last-minute bidding may be used as a form of tacit collusion. Furthermore, Bajari and Hortacsu (2003) showed that the strategy of last-minute bidding allows hiding private information about the auctioned collectors' item. Last but not least, Ariely, Ockenfels and Roth (2005) argued that it is a best reply to irrational incremental bidding.

The present paper models dynamic multi-stage auctions, which seems to be a realistic representations of auctions on eBay, given that an auction typically lasts for several days on eBay. In the proposed model, in the last stage of an auction, the transmission of a bid is uncertain in the sense that each submitted bid is rejected with some positive probability. This assumption is motivated by the fact that Internet connections are imperfect, so a bid submitted in the very last moment of the auction may simply not arrive on time. Roth and Ockenfels (2006) also acknowledged this uncertain bid transmission and introduced it in their model.

The present paper proposes a theoretical model with two private-value second-price auctions. In such setting, bidders do no wish to simply bid their valuations. If they did so, the bidder with the highest valuation could end up paying the price equal to the second highest valuation, which is too much. In an equilibrium, bidders are able to achieve the price below that, if the bidder with the highest valuation ends up in a different auction than the bidder with the second highest valuation.

Separating the bidder with the highest valuation from the bidder with the second highest valuation requires certain coordination. It is not easy at the beginning, because bidders are unable to distinguish ex-ante the strongest bidders.

Therefore, bidders may be unable to achieve this goal by simply submitting one bid and may need to submit more bids.

In order to separate the bidder with the highest valuation from the bidder with the second highest valuation, bidders wish to bid first less aggressively. In our model, these first low bids are derived from the same increasing bidding function and are used to test who value the good the most. After bidders submit their first bids, the identity of the bidders with the highest bid is revealed. Since bids were increasing in valuations, this bidder has to have the highest valuation. The test therefore reveals the identity of the bidder with the highest valuation.

After learning which bidder has the highest valuation, bidders are able to coordinate between the auctions so that the bidder with the highest valuation will end in a different auction than the bidder with the second highest valuation. The coordination requires the reallocation of certain bidders. In particular, the losing bidders need to shift to the other auction and the winning bidder needs not to move to the other auction. Then, the bidder with the second highest valuation will end in the other auction, while the bidder with the highest valuation will end in the auction where the first low bids were submitted.

There are two possibilities concerning the reallocation. If the first bids are submitted early enough, bidders may safely reallocate before the last-minute. The resulting outcome will be efficient and the final prices will equal in expectation. By contrast, if bidders send their first bids late, the reallocation has to take place in the last minute. Then, the uncertain bid transmission introduces inefficiency and causes price difference across competing auctions.

The paper is structured as follows. The next section introduces the model. Section 3 explains why bidders need first to test the ordering of their valuations and later to reallocate. It also presents two types of efficient equilibria with multiple bidding. Section 4 analyzes what happens when the reallocation takes place in the last minute. The arising inefficient equilibrium is described. Finally, section 5 concludes.

2 Elements of the model

We begin our discussion of competing Internet auctions by introducing the theoretical model capturing the rules specified by eBay. The model describes two simultaneous competing hard-close auction. The transmission of a bid is always certain except of the last minute. Bidders have unit demands.

Suppose that two second-price auctions (auction 1 and auction 2) offer the same good. Buyer i ($i = 1, \dots, N$, where $N > 2$) has an independent private valuation of one (and only one) item of the good (v_i), which is distributed according to distribution $F(v_i)$ with density $f(v_i)$ and support on $[0, 1]$. $v^{m:n}$ denotes the m^{th} highest valuation out of n bidders. $F^{m:n}(.)$ is distribution of $v^{m:n}$. $f^{m:n}(.)$ is corresponding density.

The two auctions are run simultaneously for three stages. A bidder is allowed to submit only one (non-negative) bid per stage and may only submit a bid that is higher from his previous bids and the current standing price.

Each submitted bid is accepted with certainty in the two first stages and with probability $\alpha \in (0, 1)$ in the last stage. A buyer whose bid has been accepted is registered as an active bidder.

At each stage t ($t = 1, 2, 3$) of an auction a ($a = 1, 2$), bidders first learn the current standing price (p_{at}). The current standing price in auction a is given by 0 in the first stage ($p_{a1} = 0$) and later changes according to the accepted bids. If the price is given by someone's bid, the identity of that bidder is revealed.

After the current standing price is revealed, bidders may submit non-negative bids in an auction, provided that their bids are higher than their previous bids submitted in this auction and the current standing price. Only one bid per stage is allowed and a bidder may not simultaneously bid in two auctions.

After bids are collected, the bidder with the highest accepted bid is selected as the current winner. If two highest bid are accepted in the same stage of an auction, the current winner is randomly selected among the bidders who submitted those bids. If two highest bids are accepted in different stages of the same auction, the bidder who submitted the highest bid as the first one is chosen as the current winner. Once the current winner is selected, his identity is announced and the stage ends.

After the first stage, the current standing price in auction a at stage t (p_{at}) is given by the highest bid submitted by the previous stage by the opponent of the current winner. When only one bid is submitted in an auction, the current price remains zero. Similarly, if only the current winner has been active in a given auction, the current price is given by 0 in this auction. In other words, the current winner may submit multiple bids without affecting the current price. The current price becomes positive in a given auction after at least two bidders submitted positive bids in this auction. It will be later revised, if some other buyer than the current winner submits a bid that is higher than the current standing price.

To reiterate, a stage of an auction begins with the anouncement of the price, which is followed by the announcement of the identity of the bidder whose bid is equal to the current standing price. Bidders are then given an opportunityto bid . Next, the current winner is selected. His identity is announced and the stage ends.

After the last stage the current winner in auction a becomes the final winner. He wins an object and pays the final price (p_a), which is chosen on the same basis as the current standing price. The utility of buyer i who is awarded two objects is given by: $u_i = v_i - p_1 - p_2$. If buyer i is only a final winner in auction a, his utility equals to $u_i = v_i - p_a$. Finally, a buyer who does not win any object has a zero utility.

3 Multiple-bidding without last-minute bidding

In this section, we will focus on equilibria without last-minute bidding. In the context of our model, last-minute bidding is defined as bidding in stage 3. Last-minute bids may be the only bids submitted by the bidders or alternatively last-minute bidding may occur after bidders submit their first bids. As the following example will illustrate, it does not seem rational for bidders to bid only in the last stage, beacause by doing so they would risk paying too much.

Consider for example a situation where there are only four bidders ($N = 4$) and suppose that two of them bid in the last stage of auction 1, while the other two bid in the last stage of auction 2. Since it is their last chance to win the object, it is their dominant strategy to bid their valuations. When they do so, it may happen that the bidder with the highest valuation wins the good for the price equal to the second highest valuation in the one auction and the bidder with the third highest valuation wins the good for the price equal to the fourth highest valuation. This outcome is inefficient because the bidder with the second highest valuation does not win the good. What's more, the buyer with the highest valuation finds himself in an unfortunate position. He has to pay the price equal to the second highest valuation, while the other winner pays the price equal to only the fourth highest valuation. In other words, he ends up paying too much, because he did not realize that his strongest opponent chose the same auction. Had he had this information, he would surely have shifted to the other auction. That's why, bidders wish to gather the information about the ordering of their valuations.

The above example does not only show that bidding only in the last stage may be irrational, but also indicates the bidder with the highest valuation wishes to separate from the bidder with the second highest valuation. This intuition will be very helpful to understand the construction of equilibria identified in this paper. The identified equilibria will include equilibria with and without last-minute bidding. In this section, we will focus on equilibria without last-minute bidding, which are easier to understand, given the risky bid transmission in the last stage. We will start with a symmetric equilibrium.

3.1 Symmetric equilibrium without last-minute bidding

Here, we will discuss an efficient symmetric equilibrium in which all the bidders test the ordering of their valuations by bidding according to the same increasing bidding function in stage 1 of auction 1. Afterwards, they learn which bidder has the highest valuation (because they know the identity of the current winner) and who has the second highest valuation (because the identity of the bidder whose bid is equal to the current price is revealed). Later, in stage 2, the current winner bids his valuation in auction 1, while his opponents bid their valuations in auction 2. The resuling outcome is efficient and the expected price is equal to the third highest valuation in both auctions.

To understand how an efficient symmetric equilibrium is constructed, note first that a bidder with the highest valuation needs to separate from the bidder with the second highest valuation in order to avoid the risk of paying too high a price. In stage 1, no-one has a clue which bidder values the good the most. This knowledge may be revealed when bidders submit their first bids. For example, when all the bidders bid according to the same strictly increasing bidding function in stage 1 of auction 1, they infer that the current winner has to have the highest valuation.

After learning who has the highest valuation, bidders wish to coordinate between the auctions so that the bidder with the highest valuation ends up in a different auction than the bidder with the second highest valuation. To achieve this goal, the bidder with the highest valuation may remain in the auction where he has just become the current winner (say auction 1) and all the other bidders may shift to the other auction. This coordination may be achieved, because the losing bidders realize that they would have no chance to win against the current winner in auction 1. The current winner may simply bid his valuation in stage 2 of auction 1 to discourage others from bidding in auction 1. As a result, the losing bidders would realize that there is only one object left for the competition and this object is offered for sale in auction 2. That's why, they would prefer to fiercely bid in auction 2. In particular, given the second-price rule, they would prefer to bid their valuations.

We have thus sketched a strategy profile according to which every bidder i bids according to $b(v_i)$ in stage 1 auction 1. In stage 2, the current winner bids the valuation in auction 1 and the losing bidders bid the valuations in auction 2. We have also signaled that for most of the time no bidder would have incentives to deviate from the discussed strategies. It remains to determine the optimal bidding function used in stage 1 of auction 1. Note that given the strategies defined above, the payoff of bidder i may be defined as:

$$\pi_1 + \pi_2$$

where:

- $\pi_1 \equiv Pr\left[v_i > v^{1:N-1}\right]\left(v_i - E\left[b(v^{1:N-1})|v_i > v^{1:N-1}\right]\right)$
- $\pi_2 \equiv Pr\left[v^{1:N-1} > v_i > v^{2:N-1}\right]\left(v_i - E\left[v^{2:N-1}|v^{1:N-1} > v_i > v^{2:N-1}\right]\right)$

In order to derive the optimal bidding function, we note that the expected payoff presented above is well known in the literature. In particular, it is equivalent to the payoff a bidder would obtain in two sequential second-price sealed-bid auctions[2]. To be more specific, π_1 corresponds to the payoff that the bidder would obtain when winning the good in the first out of the two sequential second-price sealed-bid auctions and π_2 is equivalent to the payoff that the bidder would obtain when winning the good in the second out the two sequential second-price sealed-bid auctions. The optimal bidding function used in stage 1 of auction 1 should be therefore equivalent to the optimal bidding function used in the first auction out of the two sequential second-price sealed-bid auctions. Recall that in the first auction out of the two sequential second-price sealed-bid auctions, it is optimal for bidder i to bid[3]:

$$b(v_i) = E[v^{3:N}|v^{2:N} = v_i] \tag{1}$$

[2]See Krishna (2002) for an overview of sequential auctions.
[3]See Krishna (2002) for a formal discussion.

Hence, by analogy, in the model presented in this paper, it is optimal to bid according to (1) in stage 1 of auction 1.

The optimal bidding function for stage 1 of auction 1 (1) has a straightforward interpretation. In particular, recall that no matter if bidder i bids in auction 1 or not, whenever he bids his valuations in auction 2 before stage 3, the probability that he wins the good in auction 2 is given by the probability that his valuation is equal to the second highest valuation off all the bidders ($Pr(v^{2:N} = v_i)$). In those circumstances, he will have to pay the price equal to the third highest valuation ($v^{3:N}$). Hence, his expected payment in auction 2 conditioned on the event that he wins in auction 2 is given by (1). Put it differently, by bidding $b(v_i)$ in stage 1 of auction 1, bidder i assures that if he wins the good in auction 1, he will pay the price at most equal to the price that he could otherwise pay in auction 2.

To reiterate, this subsection discusses an equilibrium in which after testing the ordering of their valuations in stage 1 of auction 1, the winning bidder bids the valuation in stage 2 of auction 1 and the losing bidders bid their valuations in stage 2 of auction 2. The expected final price in auction 1 is given by $b(v^{2:N})$. As shown in the appendix, in expectation this price is equal to the expected value of the third highest valuation ($E[p_1] = E[v^{3:N}]$). The final price in auction 2 is given by the third highest valuation ($v^{3:N}$). The resulting outcome is efficient, as the goods are awarded to the bidders with two highest valuations. The following proposition summarizes the above discussion.

Proposition 1. *A strategy profile in which each each buyer i:*
- *bids $b(v_i) = E[v^{3:N}|v^{2:N} = v_i]$ in stage 1 of auction 1,*
- *bids v_i in stage 2 of auction 1, if he has become a current winner,*
- *bids v_i in stage 2 of auction 2, otherwise,*
- *does not bid in stage 3,*

is a Bayesian Nash equilibrium. This equilibrium leads to an efficient outcome with the expected final prices equal to $E[v^{3:N}]$.

Proof. See appendix. □

3.2 Asymmetric equilibrium without last-minute bidding

The previous subsection discussed a symmetric equilibrium in which the bidder with the highest valuation separated from the bidder with the second highest valuation. To achieve this goal, all the bidders tested the ordering of their valuations by first bidding according to the same strictly increasing bidding function. In this subsection, we pose a question if there exists an equilibrium in which only a group of bidders test the ordering of their valuations.

To identify an equilibrium in which only some bidders test the ordering of their valuations, suppose that only $N-1$ bidders bid according to an increasing function $b(.)$ in stage 1 of auction 1 and that the remaining bidder does not bid at all in stage 1. Then, to avoid the risk of the bidder with the highest valuation ending up in the same auction that the bidder with the second highest valuation, the current winner needs to separate from the bidder whose bid is equal to the price, because it is likely that these two bidders have the two highest valuations. What's more, the bidder who did not bid in stage 1 of auction 1 needs to separate from the bidder who became the current winner in stage 1 of auction 1, because it may also happen that these two bidders have the two highest valuations. These two conditions are sufficient to assure that the bidder with the highest valuation end up in a different auction than the bidder with the second highest valuation.

To reiterate, when only $N-1$ bidders test the ordering of their valuations in stage 1 of auction 1 and one bidder does not bid at all in stage 1, the efficiency requires the current winner to separate from the bidder who is not active in stage 1 of auction 1 and the bidder whose bid is equal to the current standing price in stage 2 of auction 1. The easiest way to achieve this goal is to suppose that the current winner in auction 1 stays in auction 1, while the other bidders bid in auction 2.

Suppose then that $N-1$ bidders first bid in stage 1 of auction 1 and later all the bidders but the current winner bid in stage 2 of auction 2. Then, the current winner from stage 1 of auction 1 realizes that the other bidders might consider bidding in stage 3 of auction 1. To discourage such behavior, he prefers to bid his valuation in stage 2 of auction 1.

When the current winner from stage 1 of auction 1 bids the valuation in stage 2 of auction 1, the bidders who lost in stage 1 of auction 1 realize that they only have a chance to win the good in auction 2. Therefore, they weakly prefer to bid their valuations in stage 2 of auction 2. The bidder who was not active in stage 1 faces then a choice to compete against the second highest valuation in auction 2 or to compete against the highest valuation in auction 1, so he weakly prefers to bid his valuation in stage 2 of auction 2.

To derive an optimal bidding function for testing the ordering of the valuations of $N-1$ buyers in stage 1 of auction 1, suppose then that every buyer j other than buyer 1 and buyer N $(j = 2, \ldots, N-1)$ bids $b(v_j)$ (such that $b'(v_j) > 0$) in stage 1 of auction 1. If he becomes the current winner in stage 1 of auction 1, he bids v_j in stage 2 of auction 1. Otherwise, he bids v_j in stage 2 of auction 2. Suppose also that bidder N only bids v_N in stage 2 of auction 2. Finally, let bidder i (in this example $i = 1$) bid $b(y)$ in stage 1 of auction 1 and later bid v_i in stage 2 of auction 1, if he is a current winner in stage 1 of auction 1 and in stage 2 of auction 2, otherwise. Then, the expected payoff of bidder i is given by:

$$(N-2)\left(\int_0^y \int_0^{v_j} (v_i - b(v_j))dF^{1:N-3}(v_z)dF(v_j) + \int_y^1 \int_0^{v_i} (v_i - v_z)dF^{1:N-2}(v_z)dF(v_j)\right)$$

where the fist term describes the situation in which bidder i wins the good in auction 1 for the price of $b(v^{1:N-2})$ and the second term describes the situation in which bidder i wins the good in auction 2 for the price of $\max[v^{2:N-2}, v_j] = v^{2:N-1}$.

The first order condition implies:

$$(N-2)f(y)\left(\int_0^y (v_i - b(y))dF^{1:N-3}(v_z) - \int_0^{v_i} (v_i - v_z)dF^{1:N-2}(v_z)\right) = 0$$

Since in equilibrium $y = v_i$:

$$b(v_i) = v_i(1 - F(v_i)) + \frac{1}{F^{1:N-3}(v_i)} \int_0^{v_i} v_z dF^{1:N-2}(v_z) \tag{2}$$

By integrating by parts we find:

$$b(v_i) = v_i - \frac{1}{F^{1:N-3}(v_i)} \int_0^{v_i} F^{1:N-2}(v_z)dv_z$$

The expected final price in auction 1 ($E[p_1] = E[b(v^{2:N-1})]$) is equal to the expected value of the third highest valuation ($E[v^{3:N}]$).[4] The final price in auction 2 is given by the the third highest valuation ($v^{3:N}$). The resulting outcome is efficient, meaning that the buyer with the highest valuation and the buyer with the second highest valuation are each awarded an object. The following proposition presents the discussed equilibrium for the simplified version of the model with three bidders ($N = 3$).[5]

Proposition 2. *Suppose N=3. Take some buyer $j \in \{1, ..., N\}$. Then, a strategy profile in which buyer j bids v_j in stage 2 of auction 2 and each buyer $i \neq j$:*

- bids $b(v_i) = v_i(1 - F(v_i)) + \int_0^{v_i} v_z dF(v_z)$ in stage 1 of auction 1,

- bids v_i in stage 2 of auction 1, if he has become a current winner,

- bids v_i in stage 2 of auction 2, otherwise,

- does not bid in stage 3,

is a Bayesian Nash equilibrium. This equilibrium leads to an efficient outcome with the expected final prices equal to $E[v^{3:N}]$.

Proof. See appendix. □

4 Multiple bidding and last-minute bidding

The previous section discussed equilibria without last-minute bidding, that is without bidding in stage 3. In this section, we ask a question if there exists an equilibrium in which bidders bid in stage 3.

Recall that in section 3 we have provided an example suggesting that bidding only in the last stage may be irrational. In particular, we have analyzed a case

[4]For formal derivation see the proof of proposition 2 in the appendix.

[5]The simplification is introduced for consistency reasons. The same simplification is used to study last-minute bidding.

where four bidders bid their valuations in the last stage and have concluded that it yields a risk that the bidder with the highest valuation has to pay the price equal to the second highest valuation.

Here, we provide another reason why bidders may prefer to bid before the last stage rather than bid only in the last stage. In particular, we argue that when all the other bidders bid only in the last stage, a bidder may be better off by bidding in the second-last stage because this way he assures that his bid is accepted with certainty and may be never rejected as in the last stage.

Note that bidders who bid only in the last stage prefer to bid their valuations, because this is their last chance to win the good. By bidding their valuations they maximize the probability of winning the good at the price below their valuations without affecting the price they would have to pay. What's more, when they only bid in the last stage, they wish to split between the auctions.

Suppose then that everyone is supposed to bid the valuation in the last stage in the optimally chosen auction. Then, if some bidder decides to bid the valuation in the second-last stage of an auction in which he was supposed to bid in the last stage, the other bidders will not have time to reschedule their bids and will still bid in the last stage. It will be still their preference to bid their valuations, because by bidding more they would risk paying above their valuations and by bidding less they would risk not winning the good at the price below their valuations. Hence, they will bid their valuations in the last stage of the optimally chosen auction. As a result, the expected payoff of the bidder who decided to bid the valuation in the second-last stage will increase, because his bid will be accepted with certainty and his opponents will not become more aggressive. Therefore, as shown in the following proposition, the situation in which all the bidders wait with bidding for the last stage does not appear to be a Bayesian Nash equilibrium.

Proposition 3. *Take $M \in \{0, ..., N\}$. Then there does not exist a Bayesian Nash equilibrium in which:*

- no-one bids before stage 3,

- M bidders bid $\hat{b}(v_i)$ (s.t.$\hat{b}'(v_i) > 0$) in stage 3 of auction 1,

- N-M bidders bid $\bar{b}(v_i)$ (s.t.$\bar{b}'(v_i) > 0$) in stage 3 of auction 2.

Proof. See appendix. □

As proposition 3 shows, bidders do not find it optimal to bid only in stage 3. Similarly, they would not find it profitable to wait with bidding until stage 3, after bidding according to the same increasing bidding function in stage 1 of one auction. Having bid according to the same increasing bidding function in stage 1, they would prefer to reallocate between the auctions in stage 2 rather than in stage 3. Hence, the last-minute bidding appears to be potentially rational only if there is no bidding in stage 1 and there is bidding in stage 2.

In the following subsection we thus suppose that bidders do not bid in stage 1 and verify if there exists an equilibrium with last-minute bidding and bidding in stage 2. We begin our analysis with a symmetric case.

4.1 Bidders' preference for asymmetric strategies when bids are submitted late

Suppose that bidders do not bid in stage 1. Then, bidders may safely bid according to some increasing bidding function in stage 2 of some auction (say auction 1). Afterward, the bidder identified as the current winner in stage 2 of auction 1 will have a weak preference to bid his valuation in stage 3 of auction 1, so that all the other bidders will be discouraged from bidding in stage 3 of auction 1. In consequence, all the other bidders will weakly prefer to bid their valuations in stage 3 of auction 2.

To determine the optimal bidding function used by the bidders to test the ordering of their valuations in stage 2 of auction 1, let each bidder $j \neq i$ bid $b(v_j)$ (such that $b'(v_j)>0$) in stage 2 of auction 1. In stage 3, let the current winner bid his valuation in auction 1 and all the other bidders bid their valuations in auction 2. Finally, let bidder i behave just like every other bidder j, but let him bid $b(y)$ instead of $b(v_i)$. Then the expected payoff of bidder i is given by:

$$A + B + C$$

where:

- $A \equiv (N-1) \int_0^y \int_0^{v_j} (v_i - b(v_j)) dF^{1:N-2}(v_k) dF(v_j)$,
- $B \equiv (N-1) \sum_{z=0}^{N-3} \alpha^2 (1-\alpha)^z \int_y^1 \int_0^{v_i} (v_i - v_k) dF^{1+z:N-2}(v_k) dF(v_j)$,
- $C \equiv (N-1)\alpha(1-\alpha)^{N-2} \int_y^1 \int_0^{v_j} v_i dF^{1:N-2}(v_k) dF(v_j)$.

The first term of the expected payoff (A) corresponds to the situation where buyer i wins the good in auction 1 for the price of $b(v^{1:N-1})$. The last two terms (B and C) refer to the situation where buyer i loses in auction 1 and wins in auction 2.

The second term of the expected payoff (B) captures all the situations with positive final prices in auction 2. Recall that the probability that a bid is accepted in stage 3 of auction 2 is given by α. Since in auction 2 all the bids are submitted only in the last stage, the final price depends on this parameter. Note that B is composed of a sum of expressions depending on z. For example for $z = 1$ the corresponding expression is given by: $(N-1)\alpha^2(1-\alpha) \int_y^1 \int_0^{v_i} (v_i - v_k) dF^{2:N-2}(v_k) dF(v_j)$ and refers to the situation in which the bid equal to the second highest valuation from the group of $N-1$ bidders ($v^{2:N-1}$) is not accepted, while the bid equal to the third highest valuation from the same group ($v^{3:N-1}$) is accepted. This situation happens with probability $\alpha(1-\alpha)$. In this circumstances, bidder i wins the good if his bid is successfully submitted, which happens with probability α, and if his valuation is larger than $v^{3:N-1}$.

The third term of the expected payoff (C) captures a situation in which bidder i is the only active bidder in auction 2 and as a result wins the good for the price of 0. This extreme case happens with the probability $\alpha(1-\alpha)^{N-2}$.

The expected payoff is complex because it has to account for every possible combination of the accepted bids in stage 3 of auction 2. The number of combinations grows in N. To simplify the analysis, suppose that $N = 3$. Then, as shown in the appendix, the maximization of the expected payoff yields the following bidding function:

$$b(v_i) = v_i(1-\alpha) + \frac{\alpha^2}{F(v_i)} \int_0^{v_i} v_k dF(v_k)$$

The final price in auction 1 ($E(b(v^{2:3}))$) is thus given by:

$$\int_0^1 \left(v_i(1-\alpha) + \frac{\alpha^2}{F(v_i)} \int_0^{v_i} v_k dF(v_k) \right) dF^{2:3}(v_i)$$

or alternatively (after changing the order of integrals and rearranging) by:

$$\int_0^1 \left(6v_i(1-\alpha)F(v_i)(1-F(v_i)) + 3\alpha^2 v_i(1-F(v_i))^2\right) dF(v_i)$$

which simplifies to:

$$(1-\alpha)E[v^{2:3}] + \alpha^2[v^{3:3}]$$

Recall that in a symmetric equilibrium presented in proposition 1, the expected price was given by $[v^{3:3}]$ in both auctions. The difference between the equilibrium presented in proposition 1 and the strategy profile discussed in this subsection is that here the reallocation of the bidders is scheduled for stage 3 (and not for stage 2 as in proposition 1). Hence, postponing the reallocation from stage 2 to stage 3 increases the expected price in auction 1. In the same time, because of the risky bid transmission in the last stage, the expected price decreases in auction 2 ($E[p_2] = \alpha^2 E[v^{3:3}] < E[v^{3:3}]$).

To finish the analysis of the existence of a symmetric equilibrium with last-minute bidding, we ask a question if bidders have incentives to stick to the strategy resulting in this difference in the expected prices. The answer is no, since buyer i finds it more attractive to bid the valuation in round 2 of auction 2 than to follow the discussed strategy.[6] This way, he assures that he chooses an auction with the lowest expected price and does not risk his bid being rejected. Therefore, the strategy profile discussed in this subsection does not constitute an equilibrium. The following proposition summarizes this result.

Proposition 4. *Let there be only three bidders ($N = 3$). Then, there does not exist a Bayesian equilibrium where all the bidders bid $b(v_i)$ (such that $b'(v_i) > 0$) in stage 2 of one auction and the losing bidders bid their valuations in the other auction.*

[6] See the proof of proposition 4 for the formal derivation.

Proof. See appendix. □

We have thus demonstrated that there does not exist a symmetric equilibrium in which all the bidders first bid according to an increasing bidding function in stage 2 of one auction and later the losing bidders bid their valuations in stage 3 of the other auction. Given that there are two auctions selling the same goods, while each buyer demands only one item of the good, it is perhaps not surprising that the symmetric behavior is not always optimal. It seems natural to think that this asymmetric setting may call for asymmetric behavior.

4.2 Asymmetric equilibrium with last-minute bidding

The present subsection studies the existence of an asymmetric equilibrium with last-minute bidding. When discussing the existence of a symmetric equilibrium with last-minute bidding, we have indicated that the presence of random rejection of late bids introduces complexity to the expected payoff, when bidders decide to bid in the last stage. Therefore, for simplicity, in this subsection, we suppose that $N = 3$.

In particular, consider an asymmetric equilibrium in which two bidders bid according to the same increasing bidding function in stage 2 of one auction (say auction 1). Afterward, the winning bidder has incentives to bid the valuation in stage 3 of auction 1 in order to discourage the bidder who lost in stage 2 of auction 1 from bidding in stage 3 of auction 1. As a result, the losing bidder prefers to bid in the other auction. Since, it is his last chance to win the object, he weakly prefers to bid the valuation in stage 3 of auction 2. The remaining bidder prefers to compete against the weakest bidder of the two, so he decides to bid in auction 2. Since bids are randomly rejected in stage 3, he prefers to bid in stage 2 of auction 2. Furthermore, he has a weak preference to bid the valuation.

In order to derive the bidding function that two bidders use in stage 2 of auction 1, suppose that:

- buyer 2 bids $b(v_2)$, st. $b'(.) > 0$, in stage 2 of auction 1,
- if he loses, he bids the valuation in stage 3 of auction 2,
- otherwise, he bids the valuation in stage 3 of auction 1,
- buyer 1 behaves just as buyer 2 but bids $b(y)$ in place of bidding $b(v_2)$,
- buyer 3 bids the valuation in stage 2 of auction 2.

Then, the expected payoff of bidder 1 becomes:

$$\int_0^y (v_1 - b(v_2))dF(v_2) + \alpha \int_y^1 \int_0^{v_1} (v_1 - v_3)dF(v_3)dF(v_2)$$

where the first term corresponds to the situation in which after bidding $b(y)$, such that $y > v_2$, in stage 2 of auction 1, bidder 1 wins the good in auction 1 for the price of $b(v_2)$ and the second term relates to the event in which after bidding $b(y)$, such that $y < v_2$, in stage 2 of auction 1 and bidding the valuation in stage 3 of auction 2, bidder 1 wins the good for the price of v_3.

The first order condition implies:

$$v_1 - b(y) - \alpha \int_0^{v_1} (v_1 - v_3) dF(v_3) = 0$$

In equilibrium $y = v_i$, so that:

$$b(v_1) = v_1(1 - \alpha F(v_1)) + \alpha \int_0^{v_1} v_3 dF(v_3)$$

which after integration by parts simplifies to:

$$b(v_1) = v_1 - \alpha \int_0^{v_1} F(v_3) dv_3 \tag{3}$$

Observe that the optimal bidding function is decreasing in α, which implies that bidders are more aggressive in auction 1 when having a bid accepted in stage 3 becomes less likely. This is because when bidding in stage 2 of auction 1, bidders compare their expected payoff from winning the good in auction 1 to the expected payoff from winning the good in auction 2 after bidding the valuation in stage 3 of auction 2. Lower α decreases potential gains of wining the good in auction 2 as compared to the potential gains of winning the good in auction 1. As a result, bidders become more aggressive in auction 1.

Recall that under certain bid transmission, bidder 1 would bid

$$v_1 - \int_0^{v_1} F(v_3) dv_3 \tag{4}$$

and the resulting final price would be $E[v^{3:3}]$. Since (3) is higher than (4), the expected final price in auction 1 must be higher than $E[v^{3:3}]$. Hence, postponing the reallocation from stage 2 to stage 3 increases the final price in auction 1.

The expected final price in auction 2 is given by $E[p_2] = \alpha E[v^{3:3}]$. Intuitively, the more certain the bid transmission in the last minute, the higher the expected final price in auction 2. This price is always lower than in a corresponding efficient equilibrium presented in proposition 2 and given by $E[v^{3:3}]$. Note also that the resulting outcome will be inefficient, if buyer 3 has the lowest valuation and wins the good in auction 2. The following proposition summarizes the above discussion.

Proposition 5. *Let there be only three bidders ($N = 3$). Take some buyer $j \in \{1, 2, 3\}$. Then, a strategy profile in which buyer j bids v_j in stage 2 of auction 2 and each buyer $i \neq j$:*

- does not bid in stage 1,

- bids $b(v_i) = v_i(1 - \alpha F(v_i)) + \alpha \int_0^{v_i} v_3 dF(v_3)$ in stage 2 of auction 1,

- bids v_i in stage 3 of auction 1, if he has becomes a current winner,

- bids v_i in stage 3 of auction 2, otherwise,

is a Bayesian Nash equilibrium. This equilibrium may lead to an inefficient outcome. The expected final prices satisfy: $E[p_1] = E[b(v^{2:2})] > \alpha E[v^{3:3}] = E[p_2]$.

Proof. See appendix. □

4.3 Effects of last-minute bidding

This subsection discusses the differences between the corresponding equilibria with and without last-minute bidding. In particular, it compares an asymmetric equilibrium with last-minute bidding (presented in proposition 5) with an asymmetric equilibrium without last-minute bidding (presented in proposition 2) in order to measure the effects of last-minute bidding.

As proposition 5 shows, in the presence of last-minute bidding, the expected price in auction 1 will be higher than $E[v^{3:3}]$ and the expected price in auction 2 will be lower than $E[v^{3:3}]$. In the corresponding equilibrium without last-minute bidding, presented in proposition 2, the expected price is given by $E[v^{3:3}]$ in both auctions. Hence, auction 1 ends up with a higher price in presence of last-minute bidding while auction 2 reports lower price because of last-minute bidding.

In order to measure the effect of last-minute bidding on bidders' payoffs, let first focus on the payoff of bidder j (for simplicity say that $j = 3$), who bids only in auction 2. Note that this bidder never bids in the last minute (neither in equilibrium without last-minute bidding, nor in equilibrium with last-minute bidding). As before, suppose also that $N = 3$. Then, the difference between the expected payoff of buyer 3 in equilibrium without last-minute bidding and the one in equilibrium with last-minute bidding is given by:

$$(1 - \alpha) \int_0^{v_3} (v_3 - v_2) dF^{2:2}(v_2) - (1 - \alpha) v_3 \tag{5}$$

where the first term is explained by the fact that in absence of last-minute bidding bidder 3 has to pay the price equal to the valuation of bidder 2 more often than in presence of last-minute bidding and the second term relates to the fact that in absence of last-minute bidding bidder 3 never pays price of 0, while in presence of last-minute bidding this event happens with probability of α.

After integration by parts and rearranging, we rewrite (5) as:

$$(1 - \alpha) \int_0^{v_3} F^{2:2}(v_2) dv_2 - (1 - \alpha) v_3$$

which is equivalent to:

$$-(1-\alpha)\int_{0}^{v_3}(1-F^{2:2}(v_2))dv_2 < 0$$

Hence, bidder 3 obtains lower expected payoff in the absence of last-minute bidding. This effect of last-minute bidding is driven by the fact that by introducing random rejection of the competing bid, last-minute bidding lowers the frequency with each bidder 3 faces competition from other bidders.

We now evaluate the effect of last-minute bidding on the payoff of bidder 1.[7] The difference between the expected payoff of bidder 1 in equilibrium without last-minute bidding and the one in equilibrium with last-minute bidding is given by:

$$(1-\alpha)\int_{0}^{v_1}\int_{0}^{v_2}F(v_3)dv_3dF(v_2)+(1-\alpha)\int_{v_1}^{1}\int_{0}^{v_1}(v_1-v_3)dF(v_3)dF(v_2)>0$$

where the first term corresponds to the fact that in equilibrium without last-minute bidding bidders bid less aggressively in auction 1 and the second term is related to the fact that in absence of last-minute bidding bidder 1 have higher chances of winning the good in auction 2. Since both facts positively affect the payoff of bidder 1, the expected payoff difference between an equilibrium without last-minute bidding and an equilibrium with last-minute bidding is positive.

Hence, if bidder 1 is given to choose between an equilibrium with last-minute bidding and the one without last-minute bidding, he would opt for the one without last-minute bidding. In other words, bidder 1 would rather resign from last-minute bidding. He prefers to test the ordering of the valuations in stage 1 and submit his last bid in stage 2 rather than to test the ordering of the valuations in stage 2 and submit his last bid in stage 3. Nevertheless, bidder 1 may simply not face this choice. If buyer 2 sends his bid late (that is in stage 2 instead of stage 1), then buyer 1 is obliged to test the ordering of the valuations late (that is in stage 2) and may need to reallocate in the last-minute (that is in stage 3).

To conclude, last-minute bidding is beneficial for auction 1 and bidder 3 and unprofitable for auction 2 and bidders 1 and 2. The following corollary summarizes this results.

Corollary. *Let N=3. Then, equilibrium presented in proposition 5 yields higher expected price in auction 1 and higher expected payoff of bidder j than equilibrium presented in proposition 2. The expected price in auction 2 and the expected payoff of every buyer $i \neq j$ decreases in equilibrium presented in proposition 5 as compared to equilibrium presented in proposition 2.*

[7] Given symmetry between bidder 1 and bidder 2, the same effect applies to bidder 2.

5 Conclusion

The present paper studies the optimal bidding behavior in simultaneous Internet auctions selling identical goods. We argue that in such a setting bidders need to coordinate. In particular, they need to assure that the two strongest competitors do not en up fighting against each other in one auction, but instead split between the auctions. Since it is impossible to establish ex-ante which two bidders value the good the most, bidders need to learn who the strongest bidders are and use this information to allocate between the auctions.

We show that bidders learn the identity of the strongest competitor by sending less aggressive bids and observing the outcome. In particular, after bidders send less aggressive bids in one auction (say auction 1), one of them is chosen as the current winner. This bidder infers that he values the good the most. The other bidders infer that they do do not have the highest valuations.

After bidders submit their first low bids, separating the bidder with the highest valuation from the bidder with the second highest valuation becomes a simple task. In particular, it is sufficient that the bidder who has just become the current winner (and hence have inferred that his valuation is the highest) stays in the same auction, while the other bidders bid in the other auction.

The described coordination (achieved by first learning who is the strongest competitor and later allocating between the auctions) may take several forms. First, all the bidders do not need to first bid in the same auction. We show that there does not only exists an equilibrium where all the bidders test the ordering of the valuations, but there also exists an equilibrium where all but one bidder test the ordering of their valuations.

Second, the coordination may start early or late. If it starts early (that is if bidders test the ordering of their valuations early enough), bidders have time to safely reallocate and there is no last-minute bidding. If however bidders send their first bids late, some of them will be forced to bid in the last-minute. We show that both types of behavior are possible in an equilibrium. That is, there exists an equilibrium with last-minute bidding and an equilibrium without last-minute bidding.

In our model last-minute bidding causes inefficiency because bids sent in the last minute are assumed to be rejected with some positive probability. This feature of the model was first introduced by Roth and Ockenfels (2006) and is motivated by the fact that Internet connections are imperfect so that a bid sent seconds before the closure of an auction may not arrive on time and be therefore rejected. We also introduce this type of inefficiency to be able to verify if bidders would find it worthwhile to bid in the last-minute despite the clear disadvantage.

We identify an asymmetric equilibrium with last-minute bidding in which in the second-last stage (that is just before the last minute), all the bidders but one test the ordering of their valuations in one auctions (say auction 1), while one bidder bids in the other auction. Later, in the last stage (that is in the last-minute), the losing bidders from auction 1 move to auction 2. The identified equilibrium is very similar to an asymmetric equilibrium without last-minute bidding. In an asymmetric equilibrium without last-minute bidding, all the bidders but one bid in one auction (say auction 1) before the second-last stage. Later, in the second-last stage, the remaining bidder meet the losing bidders from auction 1 in auction 2.

We show that in asymmetric equilibrium with last-minute bidding, the auction where bidders' coordination starts (auction 1) generates higher price than in asymmetric equilibrium without last-minute bidding. On the other side, in asymmetric equilibrium with last-minute bidding, the auction to which the weaker bidders reallocate (auction 2) ends with a lower price than in asymmetric equilibrium without last-minute bidding. The bidders who test the ordering of their valuations obtain lower expected payoff in asymmetric equilibrium with last-minute bidding than in asymmetric equilibrium without last-minute bidding. The bidder who does not test the ordering of the valuations but simply bids in the other auctions has a higher expected payoff in asymmetric equilibrium with last-minute bidding than in asymmetric equilibrium without last-minute bidding.

Finally, we find that multiple symmetric auctions call for asymmetric behavior when auctions are close to end. In particular, we find that it is not in interest of all the bidders to submit their first bids in the second last stage of one auction (say auction 1). Instead, buyers prefer to split between the auctions before the risky last stage. This is because when all the bidders bid late (that is in the second last stage) in one auction, they realize that some of them will have to bid again in a risky last-minute (that is in the last stage). As a result, they prefer to be more aggressive in auction 1 in order to avoid the future risk. But then the price becomes to much inflated in auction 1 and it pays off to simply switch to auction 2. This kills incentives for symmetric coordination and encourages asymmetric coordination.

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Appendix

Proposition 1: *A strategy profile in which each each buyer i:*

- bids $b(v_i) = E[v^{3:N} | v^{2:N} = v_i]$ in stage 1 of auction 1,

- bids v_i in stage 2 of auction 1, if he has become a current winner,

- bids v_i in stage 2 of auction 2, otherwise,

- does not bid in stage 3,

is a Bayesian Nash equilibrium. This equilibrium leads to an efficient outcome with the expected final prices equal to $E[v^{3:N}]$.

Proof. Take some buyer $i \in \{1, ..., N\}$. Suppose that every other bidder $j \in \{1, ..., N\}/\{i\}$ behaves as defined in the proposition. We now verify the optimal behavior of bidder i in stage 3. If buyer i is a current winner in auction 1, then it is in his interest not to bid any longer, because he is already certain about winning the good for the final price equal to the current standing price. Therefore, as stated in the proposition, he does not bid in stage 3.

We now suppose that he is not a current winner in auction 1. Then, given that the maximal competing bid in auction 1 is given by $v^{1:N-1}$ and the maximal competing bid in auction 2 is given by $v^{2:N-1}$, bidder i weakly prefers to bid v_i in auction 2. Note that this bid will be accepted only if bidder i has not bid v_i before. Therefore, the best-response of bidder i may be defined as: bid v_i in stage 3 of auction 2, if you did not bid v_i in auction 2 before or do nothing otherwise. The strategy defined in the proposition implies that bidder i has already bid v_i in auction 2 and that in the stage 3 he does not bid, which is in line with the identified best-response.

We now verify the optimal behavior of bidder i in stage 2. Suppose first that bidder i became a current winner in stage 1 of auction 1. Then, no other bidder is going to bid in auction 1. That's why, bidder i weakly prefers to bid v_i in stage 2 of auction 1, which is in line with the strategy defined in the proposition.

Suppose now that bidder i did not become a current winner in stage 1 of auction 1. Then buyer i realizes that the maximal competing bid in auction 1 will be given by $v^{1:N-1}$, while the maximal competing bid in auction 2 will be given by $v^{2:N-1}$, so he weakly prefers to bid v_i in stage 2 of auction 2, which is again in line with the optimal behavior defined in the proposition.

We now verify the optimal behavior in stage 1. Given that the price will be given by $b(v^{2:N})$ in auction 1 and by $v^{3:N}$ in auction 2, the optimal bidding function should be equivalent to the optimal bidding function used in the first of the two sequential second-price sealed bid auction which is given by $b(v_i) = E[v^{3:N}|v^{2:N} = v_i]$.[8] This is in line with the proposition.

It remains to check if buyer i prefers to bid $b(v_i)$ in stage 1 of auction 1 rather than to bid in stage 1 of auction 2. Note that under the condition that he wins the good in auction 2, he would have to pay $E[v^{3:N}|v^{2:N} = v_i]$. But $E[v^{3:N}|v^{2:N} = v_i]$ is also the maximal price he will have to pay in auction 1. Hence, he would not find it profitable to deviate in stage 1 by bidding in auction 2.

All in all, we have just proven that the strategy profile presented in the proposition constitutes a Bayesian Nash equilibrium. The resulting outcome is efficient, because the bidder with the highest valuation and the bidder with the second highest valuation will be each awarded an object. The final price in auction 2 will be given by the third highest valuation. The expected price in auction 1 ($b(v^{2:N})$) will be given by the expected value of the third highest valuation ($E[v^{3:N}]$).[9] □

Proposition 2: *Let $N = 3$. Take some buyer $j \in \{1, ..., N\}$. Then, a strategy profile in which buyer j bids v_j in stage 2 of auction 2 and each buyer $i \neq j$:*

- bids $b(v_i)) = v_i(1 - F(v_i)) + \int_0^{v_i} v_z dF(v_z)$ in stage 1 of auction 1,

- bids v_i in stage 2 of auction 1, if he has become a current winner,

- bids v_i in stage 2 of auction 2, otherwise,

- does not bid in stage 3,

is a Bayesian Nash equilibrium. This equilibrium leads to an efficient outcome with the expected final prices equal to $E[v^{3:N}]$.

Proof. Suppose that every buyer i behaves as described in the proposition. We now derive the best response of buyer j. Buyer j knows that the maximal competing bid in auction 1 will be given by $v^{1:N-1}$, while the maximal competing bid in auction 2 will be given by $v^{2:N-1}$, so he weakly prefers to bid v_j in stage 2 of auction 2, just as indicated in the proposition.

We now derive the optimal behavior of some buyer i given that every other buyer behaves as described in the proposition. We start from stage 3. If buyer i is a current winner in some auction, then he has no interest in further bidding, because he is already certain about winning the good for the price given by the current standing price. If buyer i is not a current winner in any auction, then he knows the the highest competing bid will be given by $v^{1:N-2}$ and he believes that $v^{1:N-2} > v_i$. Hence, if on the equilibrium path buyer i is not a current winner, he either prefers to bid in stage 3 of auction 2 or not to bid at all. If he decides to bid in stage 3 of auction 2, he weakly prefers to bid the valuation. He would be allowed to bid his valuation in stage 3 of auction 2, only if he did not bid his valuation before. But on the equilibrium path he has already bid his valuation in auction 2. Hence, it is in interest of buyer i not to bid at all in stage 3.

[8] See Krishna (2002) for the formal derivation.

[9] See Krishna (2002) for the formal explication why $b(v^{2:N}) = E[v^{3:N}]$.

We now turn to stage 2 and derive the optimal behavior of buyer i. If bidder i was the current winner in stage 1 of auction 1, he is now sure about winning the good in auction 1 for the price equal to the current standing price. He has thus a weak preference to bid his valuation in stage 2 of auction 1. If he was not a current winner in stage 1 of auction 1, he infers that his valuation is below $v^{1:N-2}$. Given that the current winner will bid $v^{1:N-2}$ in stage 2 of auction 1, he weakly prefers to bid his valuation in stage 2 of auction 2.

We now turn to stage 1 and derive the optimal behavior of buyer i. In order to verify whether bidder i would prefer to bid $b(v_i)$ and later behave as presented in the proposition, we first let him bid $b(y)$ and suppose that all the other bidders behave as described in the proposition. The expected payoff of buyer i, denoted by Π, becomes:

$$\Pi = (N-2)(A+B)$$

where:

$$A \equiv \int_0^y \int_0^{v_j} \left(v_i - v_j(1-F(v_j)) - \frac{1}{F^{1:N-3}(v_j)} \int_0^{v_j} v_z dF^{1:N-2}(v_z) \right) dF^{1:N-3}(v_k) dF(v_j)$$

$$B \equiv \int_y^1 \int_0^{v_i} (v_i - v_z) dF^{1:N-2}(v_z) dF(v_j)$$

Note that:

$$\frac{\partial A}{\partial y} = f(y) \int_0^y \left(v_i - y(1-F(y)) - \frac{1}{F^{1:N-3}(y)} \int_0^y v_z dF^{1:N-2}(v_z) \right) dF^{1:N-3}(v_k)$$

$$\frac{\partial B}{\partial y} = -f(y) \int_0^{v_i} (v_i - v_z) dF^{1:N-2}(v_z)$$

Rewrite $\frac{\partial A}{\partial y}$ and $\frac{\partial B}{\partial y}$ as:

$$\frac{\partial A}{\partial y} = f(y) \left(v_i F^{1:N-3}(y) - y\left(F^{1:N-3}(y) - F^{1:N-2}(y)\right) - \int_0^y v_z dF^{1:N-2}(v_z) \right)$$

$$\frac{\partial B}{\partial y} = f(y) \left(-v_i F^{1:N-2}(v_i) + \int_0^{v_i} v_z dF^{1:N-2}(v_z) \right)$$

Substitute $y = v_i$ to find that $\frac{\partial \Pi}{\partial y} = (N-2)\left(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial y}\right) = 0$. Note also that for $y < v_i$ and $N = 3$ the first derivative becomes:

$$f(y) \left(v_i\left(1 - F^{1:N-2}(v_i)\right) - y\left(1 - F^{1:N-2}(y)\right) + \int_y^{v_i} v_z dF^{1:N-2}(v_z) \right)$$

which after integrating the last term by parts may be rewritten as:

$$f(y)\left(v_i - y - \int\limits_{y}^{v_i} F^{1:N-2}(v_z)dv_z\right)$$

or alternatively as:

$$f(y)\int\limits_{y}^{v_i}\left(1 - F^{1:N-2}(v_z)\right)dv_z > 0$$

Hence, bidder i would not find it profitable to choose $y < v_i$ in place of v_i.

When $y > v_i$ the first derivative becomes:

$$(N-2)f(y)(v_i(F^{1:N-3}(y)-F^{1:N-2}(v_i))-y(F^{1:N-3}(y)-F^{1:N-2}(y))-\int\limits_{v_i}^{y} v_z dF^{1:N-2}(v_z))$$

which after integrating the last term by parts may be rewritten as:

$$(N-2)f(y)(F^{1:N-3}(y)(v_i - y) + \int\limits_{v_i}^{y} F^{1:N-2}(v_z)dv_z)$$

or alternatively as:

$$-(N-2)f(y)\int\limits_{v_i}^{y}\left(F^{1:N-3}(y) - F^{1:N-2}(v_z)\right)dv_z$$

which is smaller than:

$$-(N-2)f(y)\int\limits_{v_i}^{y}\left(F^{1:N-3}(y) - F^{1:N-2}(y)\right)dv_z < 0$$

Hence, bidder i would not find it profitable to choose $y > v_i$ in place of v_i. All in all, if buyer i decides to bid in stage 1 of auction 1, he prefers to bid $b(v_i)) = v_i(1 - F(v_i)) + \frac{1}{F^{1:N-3}(v_i)}\int_0^{v_i} v_z dF^{1:N-2}(v_z)$.

It remains to check if buyer i prefers to bid $b(v_i)$ in stage 1 of auction 1 rather than to bid in stage 1 of auction 2. Note that once he decides not to bid in stage 1 of auction 1, he weakly prefers to bid his valuation in stage 1 of auction 2 and not to bid any longer. Hence, the situation where bidder i bids in stage 1 of auction 2 becomes payoff-equivalent to the situation where bidder i bids $b(0) = 0$ in stage 1 of auction 1 and behaves as defined in the proposition. We have already verified which bid $b(y)$ is optimal for buyer i and have shown that bidder i prefers to bid $b(v_i)$ rather than any other $b(y)$. Therefore, buyer i does not find it profitable to deviate in stage 1 by bidding in auction 2.

All in all, we have just shown that the strategy profile presented in the proposition is a Bayesian Nash equilibrium. The resulting outcome is efficient, because the bidder with the highest valuation and the bidder with the second highest valuation are each awarded an object. The final price is given by $v^{3:N}$ in auction 2.

Below we derive the expected final price in auction 1 ($E[b(v^{2:N-1})]$):

$$E[p_1] = P + Q$$

where:

$$P \equiv \int_0^1 v_i(1 - F(v_i))dF^{2:N-1}(v_i)$$

$$Q \equiv \int_0^1 \left(\int_0^{v_i} v_z dF^{1:N-2}(v_z)(N-1)(N-2)(1-F(v_i)) \right) dF(v_i)$$

Rewrite P as:

$$P = (N-1)(N-2)\int_0^1 v_i(1-F(v_i))^2 F^{N-3}(v_i)dF(v_i)$$

Change the order of integrals in Q to find:

$$Q = \frac{(N-1)(N-2)^2}{2}\int_0^1 v_z(1-F(v_z))^2 F(v_z)^{N-3}dF(v_z)$$

Hence $E[p_1]$ becomes:

$$\int_0^1 v_i \cdot \frac{N(N-1)(N-2)}{2} F(v_i)^{N-3}(1-F(v_i))^2 f(v_i)dv_i = E[v^{3:N}]$$

□

Proposition 3: *Take $M \in \{0, ..., N\}$. Then there does not exist a Bayesian Nash equilibrium in which:*

- no-one bids before stage 3,

- M bidders bid $\hat{b}(v_i)$ (s.t.$\hat{b}'(v_i) > 0$) in auction 1 in stage 3,

- N-M bidders bid $\bar{b}(v_i)$ (s.t.$\bar{b}'(v_i) > 0$) in auction 2 in stage 3.

Proof. Suppose the opposite. Without loss of generality suppose also that bidder i bids $\hat{b}(v_i)$ in stage 3 of auction 1. Then, if bidder i bids v_i in stage 2 of auction 1, all the other bidders will behave as before. What's more, his bid will be now accepted for sure and not with probability α. Hence, he will increase the probability of winning the good at the price not higher than his valuation and the values of the competing bids will remain the same. But this implies that his expected payoff will increase. A contradiction. □

Proposition 4: *Let there be only three bidders ($N = 3$). Then, there does not exist a Bayesian equilibrium where all the bidders bid $b(v_i)$ (such that $b'(v_i) > 0$) in stage 2 of one auction and the losing bidders bid their valuations in stage 3 of the other auction.*

Proof. Suppose the opposite. Let there exist a Bayesian equilibrium in which every bidder i bids in bid $b(v_i)$ (such that $b'(v_i) > 0$) in stage 2 of one auction (without loss of generalization let it be auction 1) and later let the losing bidders bid in stage 3 in the other auction (auction 2). Then the expected payoff of bidder i (denoted by Π) is given by:

$$\Pi = \pi_1 + \pi_2 + \pi_3$$

where:

$$\pi_1 \equiv 2 \int_0^y \int_0^{v_j} (v_i - b(v_j)) dF(v_k) dF(v_j)$$

$$\pi_2 \equiv 2\alpha^2 \int_y^1 \int_0^{v_i} (v_i - v_k) dF(v_k) dF(v_j)$$

$$\pi_3 \equiv 2\alpha(1-\alpha) \int_y^1 \int_0^{v_j} v_i dF(v_k) dF(v_j)$$

The first order condition implies:

$$\int_0^y (v_i - b(y)) dF(v_k) - \alpha^2 \int_0^{v_i} (v_i - v_k) dF(v_k) - \alpha(1-\alpha) \int_0^y v_i dF(v_k) = 0$$

Since in equilibrium $y = v_i$:

$$b(v_i) = v_i(1-\alpha) + \frac{\alpha^2}{F(v_i)} \int_0^{v_i} v_k dF(v_k) \tag{6}$$

The expected payoff of bidder i has to be higher than the one he would receive if he bid v_i in stage 2 of auction 2 and submitted no other bids, while his opponents behaved as described above. Note that if he follows the strategy defined in the proposition, his expected payoff is given by:

$$A + B + C$$

where:

$$A \equiv Pr[v_i > v^{1:2}] \cdot (v_i - E[b(v^{1:2})|v_i > v^{1:2}])$$
$$B \equiv \alpha^2 Pr[v^{1:2} > v_i > v^{2:2}] \cdot (v_i - E[v^{2:2}|v^{1:2} > v_i > v^{2:2}])$$

$$C \equiv \alpha(1-\alpha)Pr[v_i < v^{1:2}] \cdot v_i$$

If instead bidder i bids only v_i in stage 2 of auction 2 and his opponents behave as described above, the expected payoff of bidder i may be presented as:

$$D + E + F + G$$

where:

$$D \equiv \alpha Pr[v_i > v^{1:2}] \cdot (v_i - E[v^{2:2}|v_i > v^{1:2}])$$

$$E \equiv \alpha Pr[v^{1:2} > v_i > v^{2:2}] \cdot (v_i - E[v^{2:2}|v^{1:2} > v_i > v^{2:2}])$$

$$F \equiv (1-\alpha)Pr[v_i < v^{1:2}] \cdot v_i$$

$$G \equiv (1-\alpha)Pr[v_i > v^{1:2}] \cdot v_i$$

We observe that $B + C < E + F$. Hence, it must be that $A > D + G$. Note that given (6), A may be written as:

$$A = \int_0^{v_i} \left(v_i - (1-\alpha)v_j - \frac{\alpha^2}{F(v_j)} \int_0^{v_j} v_k dF(v_k) \right) dF^{1:2}(v_j)$$

After rearranging we find:

$$A = 2 \int_0^{v_i} \int_0^{v_j} \left(v_i - (1-\alpha)v_j - \alpha^2 v_k \right) dF(v_k) dF(v_j)$$

Since $v_k < v_j$ and $-1 + \alpha - \alpha^2 < -\alpha$:

$$A < 2 \int_0^{v_i} \int_0^{v_j} (v_i - \alpha v_k) dF(v_k) dF(v_j) = D + G$$

A contradiction. $\square$

Proposition 5: *Let there be only three bidders ($N = 3$). Take some buyer $j \in \{1, 2, 3\}$. Then, a strategy profile in which buyer j bids v_j in stage 2 of auction 2 and each buyer $i \neq j$:*

- does not bid in stage 1,

- bids $b(v_i) = v_i(1 - \alpha F(v_i)) + \alpha \int_0^{v_i} v_3 dF(v_3)$ in stage 2 of auction 1,

- bids v_i in stage 3 of auction 1, if he has become a current winner,

- bids v_i in stage 3 of auction 2, otherwise,

is a Bayesian Nash equilibrium. This equilibrium may lead to an inefficient outcome. The expected final prices satisfy: $E[p_1] = E[b(v^{2:2})] > \alpha E[v^{3:3}] = E[p_2]$.

Proof. Without loss of generality suppose that $j = 3$. Suppose also that every bidder $i \neq 3$ behaves as defined in the proposition. Then from the perspective of bidder 3 the highest competing bid will be given by $v^{1:2}$ in auction 1 and by $v^{2:2}$ in auction 2, so bidder 3 weakly prefers to bid the valuation in stage 2 of auction 2.

Without loss of generality, we now verify if bidder 1 would find it profitable to deviate from the strategy presented in the proposition. Suppose then that the other bidders behave as defined in the proposition. Then, if bidder 1 becomes a current winner in stage 2 of auction 1, it is in his interest to bid the valuation in stage 3 of auction 1, because he is already certain about winning the good in auction 1 for the price equal to the current price. If he is not a current winner in stage 2 of auction 1, he infers that $v_1 < v_2$. Furthermore, if he decides to bid in stage 3 of auction 1, it is in his interest to bid his valuation. Suppose then that he bids the valuation in stage 3 of auction 1. Then, he will win the good in auction 1, if his bid is accepted in stage 3 of auction 1, the bid of bidder 2 is rejected and $v_1 > b(v_2)$. Given that the bid of bidder 2 is given by $b(v_2) = v_2(1 - \alpha F(v_2)) + \alpha \int_0^{v_2} v_3 dF(v_3) = v_2 - \alpha \int_0^{v_2} F(v_3)dv_3$, his expected payoff may be written as:

$$\alpha(1-\alpha) \int_{v_1}^{\min[b^{-1}(v_1),1]} \left(v_1 - v_2 + \alpha \int_0^{v_2} F(v_3)dv_3 \right) dF(v_2)$$

Since $v_2 > v_1$, the above expression is smaller than:

$$\alpha^2(1-\alpha) \int_{v_1}^{\min[b^{-1}(v_1),1]} \int_0^{v_2} F(v_3)dv_3 dF(v_2)$$

By rearranging the order of integrals, the above expression becomes equivalent to:

$$\alpha^2(1-\alpha) \int_0^{v_1} \int_{v_3}^{\min[b^{-1}(v_1),1]} dF(v_2)F(v_3)dv_3$$

or alternatively to:

$$\alpha^2(1-\alpha) \int_0^{v_1} (F(\min[b^{-1}(v_1),1]) - F(v_3))F(v_3)dv_3$$

Since $(F(\min[b^{-1}(v_1),1]) - F(v_3) < 1$ and $\alpha^2(1-\alpha) < \alpha$, the above expression is smaller than:

$$\alpha \int_0^{v_1} F(v_3)dv_3$$

or equivalently than:

$$\alpha \int_0^{v_1} (v_1 - v_3) dF(v_3)$$

which is equal to the payoff that bidder 1 would obtain if he bid the valuation in stage 3 of auction 2. Note that given the second-price rule, it is his weak preference to bid his valuation in stage 3 of auction 2 rather to submit any other bid in stage 3 of auction 2. Hence, overall, when bidder 1 does not become the current winner in stage 2 of auction 1, he prefers to bid the valuation in stage 3 of auction 2, just like dictated by the strategy defined in the proposition.

We now analyze the optimal behavior of bidder 1 in stage 2. Note first that bidder 1 prefers to bid in stage 2 of auction 1 rather than in stage 2 of auction 2, because the highest competing bid in auction 1 will be given by $E[b(v_2)] < E[v_2] = E[v_3]$ and the highest competing bid in auction 2 will be given by $E[v_3]$. Suppose then that he bids $b(y)$ in stage 2 of auction 1. Then his expected payoff is given by:

$$\int_0^{y} \left(v_1 - v_2(1 - \alpha F(v_2)) - \alpha \int_0^{v_2} v_3 dF(v_3) \right) dF(v_2) + \alpha \int_y^{1} \int_0^{v_1} (v_1 - v_3) dF(v_3) dF(v_2)$$

The first order condition implies:

$$f(y) \left(v_1 - y(1 - \alpha F(y)) - \alpha \int_0^{y} v_3 dF(v_3) - \alpha \int_0^{v_1} (v_1 - v_3) dF(v_3) \right) = 0$$

which simplifies to:

$$f(y) \left(v_1 - y + \alpha F(y) y - \alpha F(v_1) v_1 - \alpha \int_0^{y} v_3 dF(v_3) + \alpha \int_0^{v_1} v_3 dF(v_3) \right) = 0$$

and solves for $y = v_1$. Note that for $y < v_1$ the first derivative becomes:

$$f(y) \left(v_1 - y + \alpha F(y) y - \alpha F(v_1) v_1 + \alpha \int_y^{v_1} v_3 dF(v_3) \right)$$

which after integration by parts may be rewritten as:

$$f(y) \left(v_1 - y - \alpha \int_y^{v_1} F(v_3) dv_3 \right)$$

or alternatively as:

$$f(y) \int_y^{v_1} (1 - \alpha F(v_3)) dv_3$$

which is higher than 0. Hence, bidder 1 would not find it profitable to choose $y < v_1$ in place of v_1.

When $y > v_1$ the first derivative becomes:

$$f(y)\left(v_1 - y + \alpha F(y)y - \alpha F(v_1)v_1 - \alpha \int_{v_1}^{y} v_3 dF(v_3)\right)$$

which after integration by parts may be rewritten as:

$$f(y)\left(v_1 - y + \alpha \int_{v_1}^{y} F(v_3)dv_3\right)$$

or alternatively as:

$$-f(y)\int_{v_1}^{y}(1 - \alpha F(v_3))dv_3$$

which is smaller than 0. Hence, bidder 1 would not find it profitable to choose $y > v_1$ in place of v_1.

Finally, note that it is impossible for bidder 1 to profitably deviate by bidding in stage 1, as his opponents would not change their bids then.

Thus, the strategy profile presented in the proposition is a Bayesian Nash equilibrium. The resulting outcome may be inefficient, because it may happen that having the lowest valuation, bidder 3 wins the good. The expected final price is given by $\alpha E[v^{3:3}]$ in auction 2. Below we derive the expected final price in auction 1 ($E[b(v^{2:2})]$):

$$E[b(v^{2:2})] = P + Q$$

where:

$$P \equiv 2\int_{0}^{1} v_i(1 - \alpha F(v_i))(1 - F(v_i))f(v_i)dv_i$$

$$Q \equiv 2\int_{0}^{1}\int_{0}^{v_i} \alpha v_z dF(v_z)(1 - F(v_i))f(v_i)dv_i$$

After changing changing the order of integrates, reformulate Q as:

$$Q = \alpha \int_{0}^{1} v_z(1 - F(v_z))^2 f(v_z)dv_z$$

Note that:

$$P > 2\alpha \int_{0}^{1} v_i(1 - F(v_i))^2 f(v_i)dv_i$$

Hence:

$$P + Q > 3\alpha \int_0^1 v_i(1 - F(v_i))^2 f(v_i) dv_i = \alpha E[v^{3:N}]$$

which proves that $E[p_1] = E[b(v^{2:2})] > \alpha E[v^{3:3}]$. $\square$